MATHAI_11_HL_Summer_2021_Q1

Solution

1. Define the parameters of the distribution

The problem states that the number of fish caught per hour follows a **Poisson distribution**. Let X be the random variable representing the number of fish caught in an 8-hour period.

- The mean rate per hour is $\lambda_{\text{hour}} = 1.1$.
- For a duration of $t = 8$ hours, the mean parameter λ for the distribution of X is calculated as:

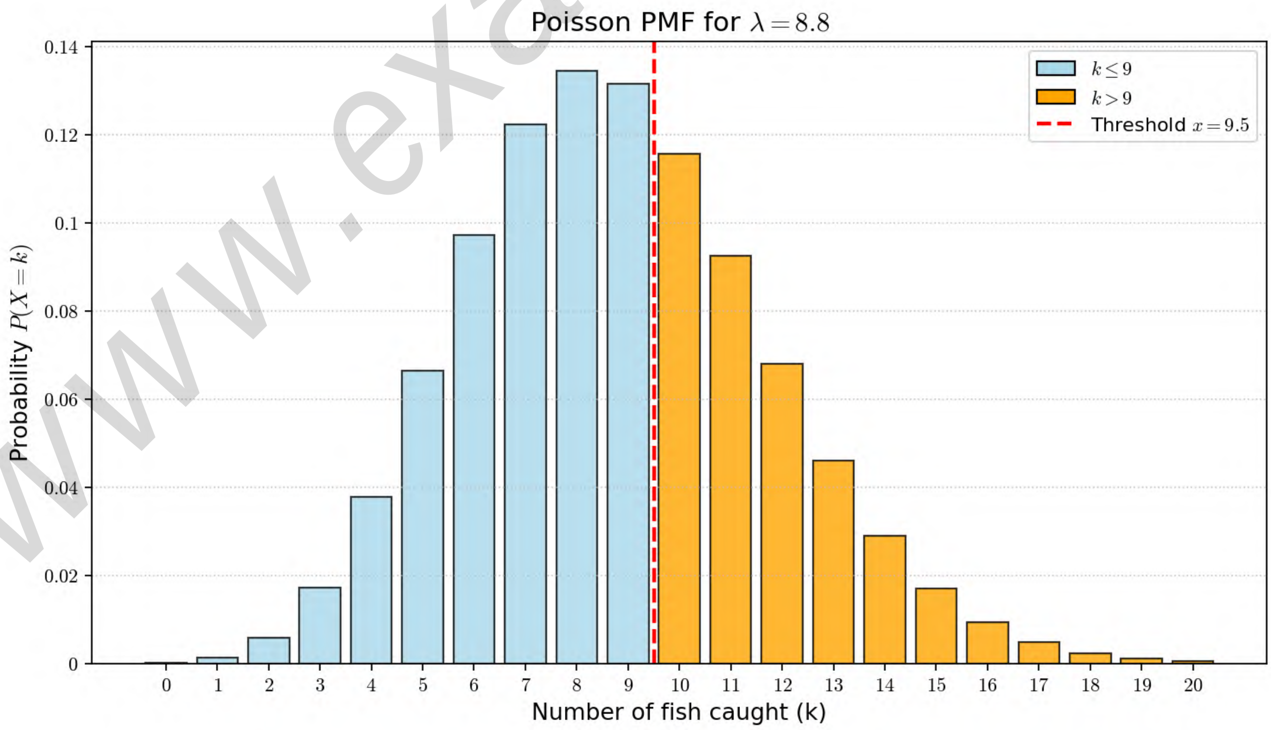
$$\begin{aligned} \lambda &= \lambda_{\text{hour}} \times t \\ &= 1.1 \times 8 \\ &= 8.8 \end{aligned}$$

Thus, $X \sim \text{Po}\,(8.8)$.

2. Formulate the probability expression

We need to find the probability that George catches more than 9 fish, which is $P(X > 9)$. Since the Poisson distribution is discrete, this is equivalent to:

$$\begin{aligned} P(X > 9) &= 1 - P(X \leq 9) \\ &= 1 - \sum_{k=0}^{9} \frac{e^{-\lambda}\lambda^k}{k!} \end{aligned}$$



3. Calculate the cumulative probability

We evaluate the sum $P(X \leq 9) = \sum_{k=0}^{9} \frac{e^{-8.8}(8.8)^k}{k!}$:

- $k = 0 : \frac{e^{-8.8} \cdot 8.8^0}{0!} \approx 0.0001507$
- $k = 1 : \frac{e^{-8.8} \cdot 8.8^1}{1!} \approx 0.0013264$
- $k = 2 : \frac{e^{-8.8} \cdot 8.8^2}{2!} \approx 0.0058363$
- $k = 3 : \frac{e^{-8.8} \cdot 8.8^3}{3!} \approx 0.0171198$
- $k = 4 : \frac{e^{-8.8} \cdot 8.8^4}{4!} \approx 0.0376635$
- $k = 5 : \frac{e^{-8.8} \cdot 8.8^5}{5!} \approx 0.0662878$
- $k = 6 : \frac{e^{-8.8} \cdot 8.8^6}{6!} \approx 0.0972221$
- $k = 7 : \frac{e^{-8.8} \cdot 8.8^7}{7!} \approx 0.1222221$
- $k = 8 : \frac{e^{-8.8} \cdot 8.8^8}{8!} \approx 0.1344443$
- $k = 9 : \frac{e^{-8.8} \cdot 8.8^9}{9!} \approx 0.1314566$

Summing these values:

$$P(X \leq 9) \approx 0.6177296$$

4. Final result

Subtract the cumulative probability from 1:

$$\begin{aligned} P(X > 9) &= 1 - 0.6177296 \\ &\approx 0.3822704 \end{aligned}$$

Rounding to four decimal places:

$$\boxed{0.3823}$$

MATHAI_11_HL_Summer_2021_Q2

Solution

Based on the principles of **Voronoi diagrams** and coordinate geometry, the solution is as follows:

1. Signal Strength Explanation

A **Voronoi diagram** partitions a plane into regions based on the distance to specific points (sites). Each region contains all points that are closer to its corresponding site than to any other site.

- Since Tim is standing inside the shaded region, which is the Voronoi cell for tower $T4$, he is geographically closer to $T4$ than to $T1$, $T2$, or $T3$.
- Assuming signal strength decreases as distance increases, Tim will receive the strongest signal from the nearest tower, which is $T4$.

2. Coordinates of Tower T4

The dashed line segment connecting vertices A and B is an edge of the Voronoi diagram. By definition, a Voronoi edge is the **perpendicular bisector** of the line segment connecting the two adjacent sites.

- The edge AB lies between the regions for tower $T2$ and tower $T4$.
- We are given that the equation of the edge AB is $y = 3$ and the coordinates of $T2$ are $(-9, 5)$.
- Since the edge is a horizontal line ($y = 3$), the line connecting $T2$ and $T4$ must be vertical (perpendicular to the edge). Thus, $T4$ must have the same x-coordinate as $T2$: $x_{T4} = -9$.
- The edge $y = 3$ must be the midpoint of the y-coordinates of $T2$ and $T4$:

$$\frac{y_{T2} + y_{T4}}{2} = 3$$
$$\frac{5 + y_{T4}}{2} = 3$$
$$5 + y_{T4} = 6$$
$$y_{T4} = 1$$

The coordinates of tower $T4$ are $\boxed{(-9, 1)}$.

3. Gradient of the Edge between T1 and T2

The edge between $T1$ and $T2$ is the perpendicular bisector of the segment $T1T2$.

- First, calculate the gradient (m) of the line segment connecting $T1(-13, 3)$ and $T2(-9, 5)$:

$$m_{T1T2} = \frac{y_2 - y_1}{x_2 - x_1}$$
$$= \frac{5-3}{-9-(-13)}$$
$$= \frac{2}{4}$$
$$= 0.5$$

- The gradient of the Voronoi edge (m_{edge}) is the **negative reciprocal** of the gradient of the segment $T1T2$:

$$m_{\text{edge}} = -\frac{1}{m_{T1T2}}$$
$$= -\frac{1}{0.5}$$
$$= -2$$

The gradient of the edge between towers $T1$ and $T2$ is $\boxed{-2}$.

MATHAI_11_HL_Summer_2021_Q3

Solution

1. Analysis of Charlie's Running Distance

Charlie's daily distance increases by a constant amount each day. This behavior is modeled by an **arithmetic progression** (AP).

- First term $a = 500\,\mathrm{m}$
- Common difference $d = 100\,\mathrm{m}$
- The general term for an AP is $u_n = a + (n-1)d$.

For day $n = 20$:

$$\begin{aligned} u_{20} &= 500 + (20-1) \times 100 \\ &= 500 + 19 \times 100 \\ &= 500 + 1900 \\ &= 2400\,\mathrm{m} \end{aligned}$$

2. Analysis of Daniella's Running Distance

Daniella's daily distance increases by a fixed percentage each day. This behavior is modeled by a **geometric progression** (GP).

- First term $a = 500\,\mathrm{m}$
- Common ratio $r = 1 + \frac{2}{100} = 1.02$
- The general term for a GP is $v_n = a \cdot r^{n-1}$.

For day $n = 20$:

$$\begin{aligned} v_{20} &= 500 \times (1.02)^{20-1} \\ &= 500 \times (1.02)^{19} \\ &\approx 500 \times 1.456811 \\ &\approx 728.4055\,\mathrm{m} \end{aligned}$$

3. Determining the Day n where Daniella runs further than Charlie

We seek the smallest integer n such that $v_n > u_n$. Substituting the general formulas:

$$500 \times (1.02)^{n-1} > 500 + (n-1) \times 100$$

Let $k = n - 1$:

$$500(1.02)^k > 500 + 100k$$

Dividing by 100:

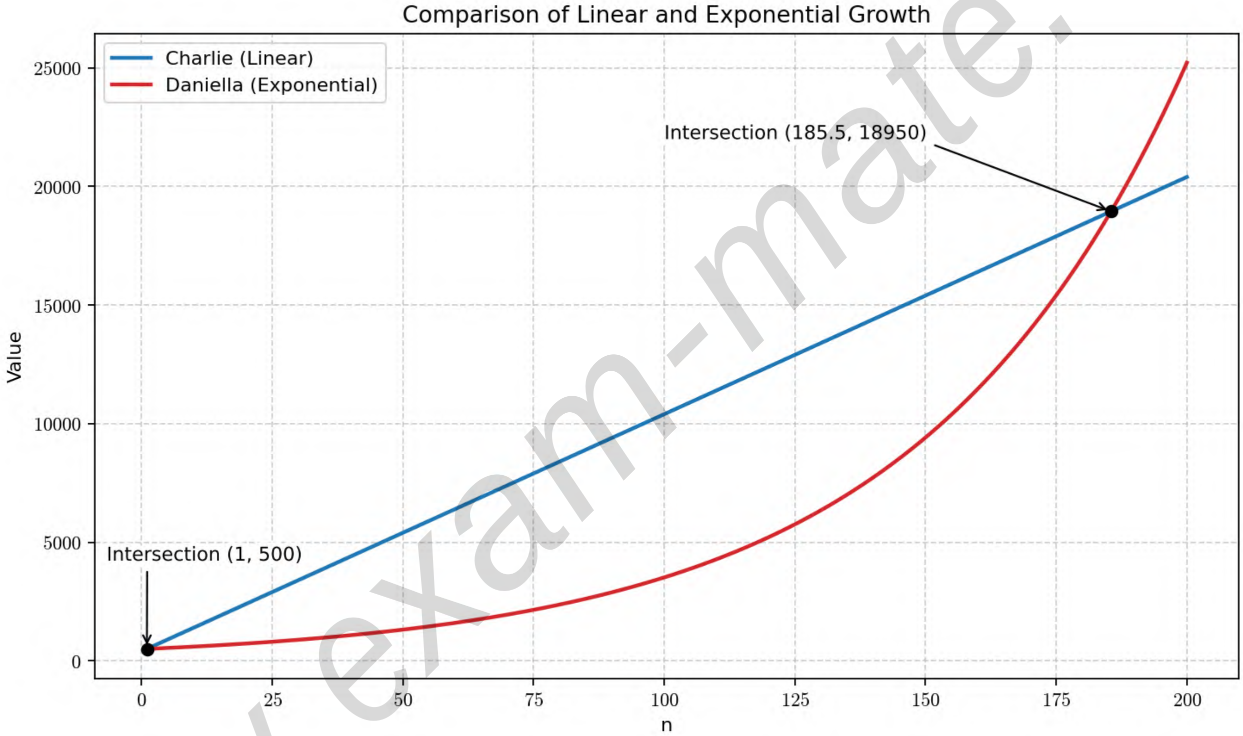
$$5(1.02)^k > 5 + k$$

We evaluate this inequality for various values of n:

- For $n = 100$ ($k = 99$):

- $u_{100} = 500 + 99 \times 100 = 10400$
- $v_{100} = 500 \times (1.02)^{99} \approx 3545.24$ ($u_{100} > v_{100}$)

- For $n = 180$ ($k = 179$):
  - $u_{180} = 500 + 179 \times 100 = 18400$
  - $v_{180} = 500 \times (1.02)^{179} \approx 17042.42$ ($u_{180} > v_{180}$)
- For $n = 184$ ($k = 183$):
  - $u_{184} = 500 + 183 \times 100 = 18800$
  - $v_{184} = 500 \times (1.02)^{183} \approx 18447.41$ ($u_{184} > v_{184}$)
- For $n = 185$ ($k = 184$):
  - $u_{185} = 500 + 184 \times 100 = 18900$
  - $v_{185} = 500 \times (1.02)^{184} \approx 18816.36$ ($u_{185} > v_{185}$)
- For $n = 186$ ($k = 185$):
  - $u_{186} = 500 + 185 \times 100 = 19000$
  - $v_{186} = 500 \times (1.02)^{185} \approx 19192.69$ ($v_{186} > u_{186}$)



Testing the boundary: At $n = 185$, $v_{185} \approx 18816.36 < 18900$. At $n = 186$, $v_{186} \approx 19192.69 > 19000$.

Final Answers: (a) (i) $\boxed{2400\,\mathrm{m}}$ (a) (ii) $\boxed{728\,\mathrm{m}}$ (to 3 s.f.) (b) $\boxed{n = 186}$

MATHAI_11_HL_Summer_2021_Q4

Solution

The problem involves an **exponential decay** model for information retention. The percentage of information retained, R, is given by the function:

$$R(t) = 100e^{-pt}, \quad t \geq 0$$

where t is the time in days and p is a constant.

1. Finding the value of p

- The problem states that 1 day after the lecture ($t = 1$), students had forgotten 50% of the information.
- If 50% is forgotten, then the percentage retained is $R(1) = 100 - 50 = 50$.
- Substitute these values into the model:

$$\begin{aligned} 50 &= 100e^{-p(1)} \\ 0.5 &= e^{-p} \\ \ln(0.5) &= -p \\ p &= -\ln(0.5) \\ p &= \ln(2) \end{aligned}$$

- Using the numerical value:

$$p \approx 0.693147180559945$$

2. Percentage of information retained after 36 hours

- First, convert the time from hours to days:

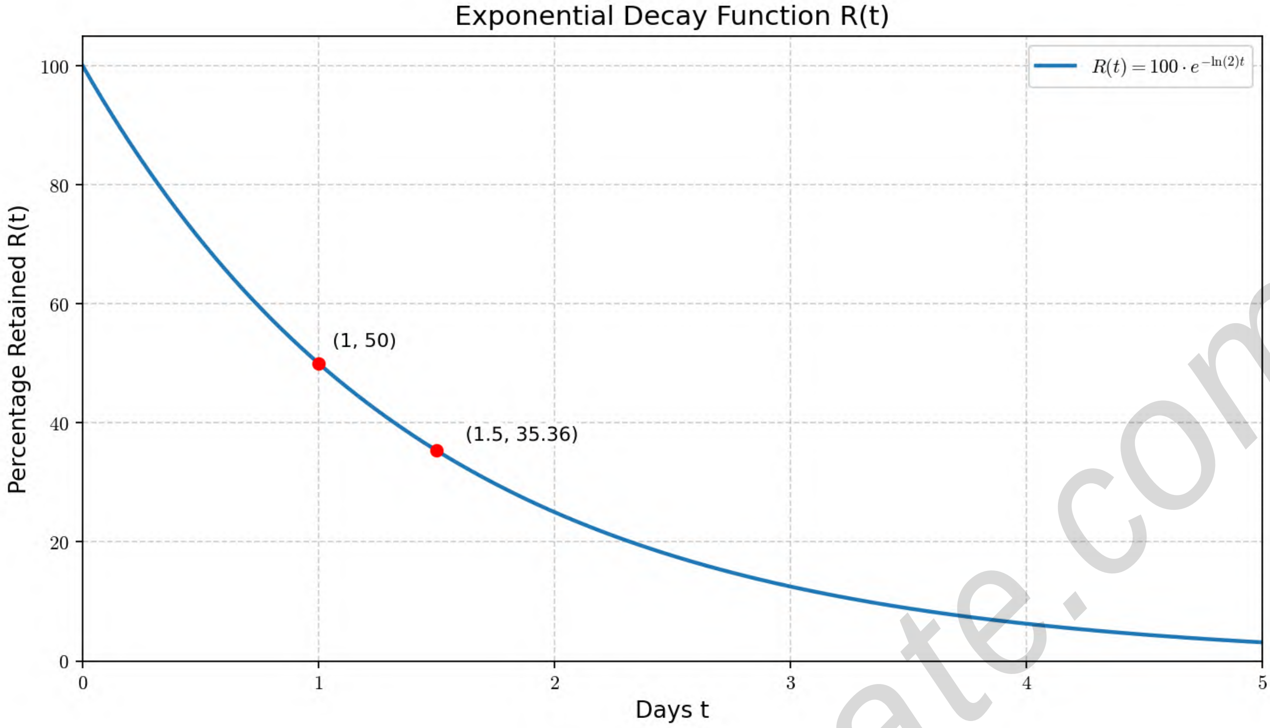
$$t = \frac{36\,\text{hours}}{24\,\text{hours/day}} = 1.5\,\text{days}$$

- Substitute $t = 1.5$ and $p = \ln(2)$ into the retention function:

$$\begin{aligned} R(1.5) &= 100e^{-\ln(2)\cdot 1.5} \\ &= 100(e^{\ln(2)})^{-1.5} \\ &= 100(2)^{-1.5} \\ &= \frac{100}{2^{1.5}} \\ &= \frac{100}{2\sqrt{2}} \\ &= \frac{50}{\sqrt{2}} = 25\sqrt{2} \end{aligned}$$

- Calculating the decimal value:

$$R(1.5) \approx 35.3553390593274$$



3. Mathematical reason for continuous retention

- The function $R(t) = 100e^{-pt}$ is an **exponential function**.
- For any finite value of t, $e^{-pt} > 0$.
- Mathematically, the **horizontal asymptote** of the function as $t \to \infty$ is $R = 0$, but the function itself never reaches zero. Therefore, $R(t) > 0$ for all $t \geq 0$.

4. Limitation of the domain

- The **domain** is given as $t \geq 0$, implying the model is valid forever.
- A possible limitation is that the model may not be accurate for very large values of t (e.g., several years later), as human memory might eventually reach a point where the information is completely lost or the percentage becomes so small it is negligible/zero in a practical sense.
- Additionally, the model does not account for **reinforcement** or restudy, which would change the retention rate over time.

(a) $\boxed{p = \ln(2) \approx 0.693}$

(b) $\boxed{35.4\%}$ (to 3 sig. figs.)

(c) The function $R(t) = 100e^{-pt}$ is always greater than zero for all finite t because the range of an exponential function a^x (where $a > 0$) is $(0, \infty)$.

(d) The model assumes the information is never completely forgotten ($R \to 0$ only as $t \to \infty$), which is unrealistic over a very long lifespan.

MATHAI_11_HL_Summer_2021_Q5

Solution

1. Determination of Vectors $\overrightarrow{CA}$ and $\overrightarrow{CB}$

To find the vectors originating from point $C(5,4,3)$ to points $A(2,0,2)$ and $B(8,0,2)$, we subtract the coordinates of the initial point from the coordinates of the terminal point.

- (i) For the vector $\overrightarrow{CA}$:

$$\begin{aligned}\overrightarrow{CA} &= \vec{A} - \vec{C}\\ &= \begin{pmatrix}2\\0\\2\end{pmatrix} - \begin{pmatrix}5\\4\\3\end{pmatrix}\\ &= \begin{pmatrix}-3\\-4\\-1\end{pmatrix}\end{aligned}$$

- (ii) For the vector $\overrightarrow{CB}$:

$$\begin{aligned}\overrightarrow{CB} &= \vec{B} - \vec{C}\\ &= \begin{pmatrix}8\\0\\2\end{pmatrix} - \begin{pmatrix}5\\4\\3\end{pmatrix}\\ &= \begin{pmatrix}3\\-4\\-1\end{pmatrix}\end{aligned}$$

2. Calculation of the Cross Product $\overrightarrow{CA} \times \overrightarrow{CB}$

The **cross product** of two vectors in $\mathbb{R}^3$ is calculated using the determinant of a 3×3 matrix involving the standard basis vectors $\mathbf{i}, \mathbf{j}, \mathbf{k}$:

$$\begin{aligned}\overrightarrow{CA} \times \overrightarrow{CB} &= \begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ -3 & -4 & -1\\ 3 & -4 & -1\end{vmatrix}\\ &= \mathbf{i}\begin{vmatrix}-4 & -1\\ -4 & -1\end{vmatrix} - \mathbf{j}\begin{vmatrix}-3 & -1\\ 3 & -1\end{vmatrix} + \mathbf{k}\begin{vmatrix}-3 & -4\\ 3 & -4\end{vmatrix}\\ &= \mathbf{i}((-4)(-1) - (-1)(-4)) - \mathbf{j}((-3)(-1) - (-1)(3)) + \mathbf{k}((-3)(-4) - (-4)(3))\\ &= \mathbf{i}(4-4) - \mathbf{j}(3+3) + \mathbf{k}(12+12)\\ &= 0\mathbf{i} - 6\mathbf{j} + 24\mathbf{k}\\ &= \begin{pmatrix}0\\-6\\24\end{pmatrix}\end{aligned}$$

3. Area of Triangle ABC

The area of a triangle defined by two vectors $\overrightarrow{CA}$ and $\overrightarrow{CB}$ is equal to half the **magnitude** of their cross product:

$$\begin{aligned}\text{Area} &= \frac{1}{2}|\overrightarrow{CA} \times \overrightarrow{CB}|\\ &= \frac{1}{2}\sqrt{0^2 + (-6)^2 + 24^2}\\ &= \frac{1}{2}\sqrt{0 + 36 + 576}\\ &= \frac{1}{2}\sqrt{612}\end{aligned}$$

Simplifying the radical:

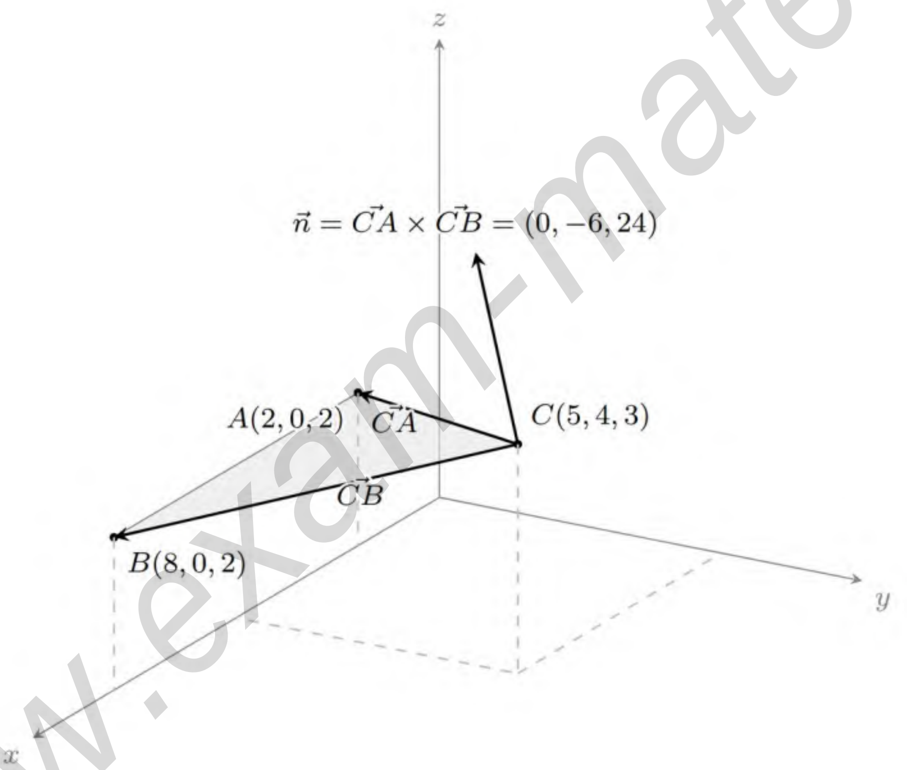
$$\sqrt{612} = \sqrt{36 \times 17}$$
$$= 6\sqrt{17}$$

Thus, the area is:

$$\text{Area} = \frac{1}{2}(6\sqrt{17})$$
$$= 3\sqrt{17}$$

Using the numerical value $\sqrt{17} \approx 4.1231$:

$$\text{Area} \approx 12.3693\,\text{m}^2$$



(a)

(i) $\overrightarrow{CA} = \begin{pmatrix} -3 \\ -4 \\ -1 \end{pmatrix}$, (ii) $\overrightarrow{CB} = \begin{pmatrix} 3 \\ -4 \\ -1 \end{pmatrix}$

(b) $\overrightarrow{CA} \times \overrightarrow{CB} = \begin{pmatrix} 0 \\ -6 \\ 24 \end{pmatrix}$

(c) Area $= 3\sqrt{17} \approx 12.37\,\text{m}^2$

MATHAI_11_HL_Summer_2021_Q6

Solution

1. Determine the upper bounds for the given measurements

To find the maximum possible area, we must identify the **upper bound** for each measurement based on the specified precision.

- For the side lengths AB and BC, the measurements are correct to the nearest metre. This implies an **absolute error** of $0.5\,\mathrm{m}$.
  - $AB_{\max} = 56 + 0.5 = 56.5\,\mathrm{m}$
  - $BC_{\max} = 82 + 0.5 = 82.5\,\mathrm{m}$
- For the angle $\angle B$, the measurement is correct to the nearest 5°. This implies an absolute error of 2.5°.
  - $\angle B_{\max} = 105 + 2.5 = 107.5^\circ$

2. Calculate the maximum area

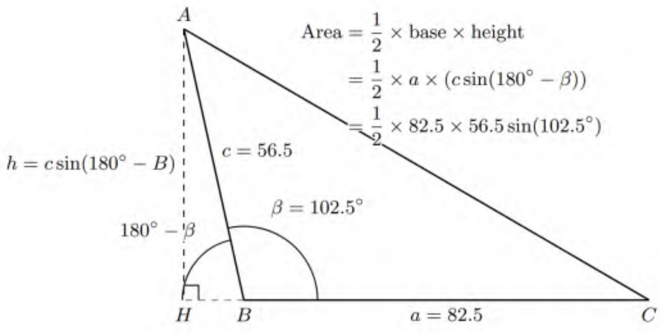
The area A of a triangle with two sides a, b and the included angle θ is given by the **SAS area formula**:

$$A = \frac{1}{2}ab\sin(\theta)$$

To maximize A, we use the maximum values for AB, BC, and $\sin(\angle B)$. Since the **sine function** is increasing for angles between 0° and 90° and decreasing between 90° and 180°, we must check if the maximum angle 107.5° yields a larger sine value than the lower bound 102.5°. Since 107.5° is further from 90° than 102.5°, we should verify which bound maximizes the sine. However, for angles in the second quadrant, $\sin(\theta)$ decreases as θ increases beyond 90°.

- $\sin(102.5^\circ) \approx 0.9763$
- $\sin(107.5^\circ) \approx 0.9537$

Wait, to maximize the area, we need the maximum value of the product $AB \cdot BC \cdot \sin(\angle B)$. While AB and BC are maximized at their upper bounds, $\sin(\angle B)$ is maximized when the angle is closest to 90° within the allowed range $[102.5^\circ, 107.5^\circ]$. Thus, the maximum value for the sine component occurs at the **lower bound** of the angle, $\angle B = 102.5^\circ$.



$$A_{\max} = \frac{1}{2} \times AB_{\max} \times BC_{\max} \times \sin(102.5^\circ)$$
$$= \frac{1}{2} \times 56.5 \times 82.5 \times \sin(102.5^\circ)$$
$$= 2330.625 \times \sin(102.5^\circ)$$
$$\approx 2275.377\ldots$$

Rounding to a standard level of precision (e.g., 1 decimal place):

$$\boxed{2275.4\,\mathrm{m}^2}$$

MATHAI_11_HL_Summer_2021_Q7

Solution

1. Area of the top of the gift box

- (i) The top of the gift box is the large right-angled triangle $\triangle GIK$. The area A of a triangle is given by $\frac{1}{2} \times \text{base} \times \text{height}$. In this right-angled triangle, the legs are GI and IK.
  - The length $GI = GH + HI$. Since $HIJL$ is a rectangle, $HI = LJ = 6\,\text{cm}$. Thus, $GI = p + 6$.
  - The length $IK = IJ + JK$. Since $HIJL$ is a rectangle, $IJ = HL = 8\,\text{cm}$. Thus, $IK = 8 + q$.
  - The area A is:

$$A = \frac{1}{2}(p+6)(q+8)$$

$$\boxed{A = \frac{1}{2}(p+6)(q+8)}$$

- (ii) To show the expression for A in terms of q only, we use **similar triangles**.
  - Note that $\triangle GHL$ and $\triangle LJK$ are both right-angled triangles (as $HIJL$ is a rectangle and $G - L - K$ is a straight line). Furthermore, since $\triangle GIK$ is right-angled at I, $\angle G + \angle K = 90^\circ$. In $\triangle GHL$, $\angle G + \angle GLH = 90^\circ$, and in $\triangle LJK$, $\angle K + \angle JLK = 90^\circ$. This implies $\triangle GHL \sim \triangle LJK$.
  - From the similarity ratio:

$$\frac{GH}{LJ} = \frac{HL}{JK}$$
$$\frac{p}{6} = \frac{8}{q}$$
$$p = \frac{48}{q}$$

  - Substitute $p = \frac{48}{q}$ into the area formula:

$$\begin{aligned} A &= \frac{1}{2}\left(\frac{48}{q} + 6\right)(q+8) \\ &= \frac{1}{2}\left(48 + \frac{384}{q} + 6q + 48\right) \\ &= \frac{1}{2}\left(96 + \frac{384}{q} + 6q\right) \\ &= 48 + \frac{192}{q} + 3q \end{aligned}$$

  Thus, $A = \frac{192}{q} + 3q + 48$.

2. Differentiation

- (b) To find the derivative of A with respect to q:

$$\frac{dA}{dq} = \frac{d}{dq}(192q^{-1} + 3q + 48)$$
$$= -192q^{-2} + 3$$
$$= 3 - \frac{192}{q^2}$$

$$\boxed{\frac{dA}{dq} = 3 - \frac{192}{q^2}}$$

3. Minimization of the area

- (c) (i) To find the value of q that minimizes the area, we set the first derivative to zero (the **stationary point** condition):

$$\frac{dA}{dq} = 0$$
$$3 - \frac{192}{q^2} = 0$$

$$\boxed{3 - \frac{192}{q^2} = 0}$$

- (ii) Solving the equation for q:

$$3 = \frac{192}{q^2}$$
$$q^2 = \frac{192}{3}$$
$$q^2 = 64$$
$$q = \pm 8$$

Since q represents a physical length, we take the positive root $q = 8$. To verify this is a minimum, we check the second derivative: $\frac{d^2A}{dq^2} = \frac{384}{q^3}$. For $q = 8$, $\frac{d^2A}{dq^2} > 0$, confirming a local minimum.

$$\boxed{q = 8}$$

MATHAI_11_HL_Summer_2021_Q8

Solution

The problem describes a game where two unbiased dice are rolled. Let D_1 and D_2 be the results of the first and second die, respectively. The score T is defined as the maximum value of the two dice: $T = \max(D_1, D_2)$. Since each die has 6 faces, there are $6 \times 6 = 36$ equally likely outcomes in the **sample space**.

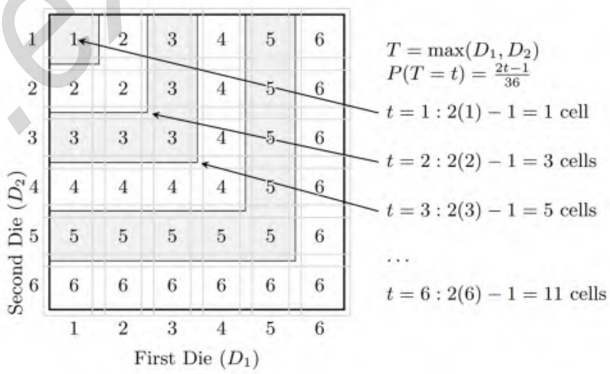
1. Probability Distribution of T

To find $P(T = t)$, we count the number of pairs (D_1, D_2) such that $\max(D_1, D_2) = t$.

- For $T = 1$: Only $(1, 1)$ works. (1 outcome)
- For $T = 2$: Outcomes are $(1, 2), (2, 2), (2, 1)$. (3 outcomes)
- For $T = 3$: Outcomes are $(1, 3), (2, 3), (3, 3), (3, 2), (3, 1)$. (5 outcomes)
- In general, for a score t, the outcomes are $(1, t), ..., (t - 1, t), (t, t), (t, t - 1), ..., (t, 1)$. The number of outcomes is $2t - 1$.

The **probability distribution** table is:

t	1	2	3	4	5	6
$P(T = t)$	$\frac{1}{36}$	$\frac{3}{36}$	$\frac{5}{36}$	$\frac{7}{36}$	$\frac{9}{36}$	$\frac{11}{36}$



2. Probabilities for Specific Events

- **(i) Probability that a player scores at least 3:** This is the sum of probabilities for $T \in \{3, 4, 5, 6\}$.

$$\begin{aligned}P(T \geq 3) &= P(T=3)+P(T=4)+P(T=5)+P(T=6)\\ &= \frac{5}{36}+\frac{7}{36}+\frac{9}{36}+\frac{11}{36}\\ &= \frac{32}{36}\\ &= \frac{8}{9}\end{aligned}$$

Alternatively, using the **complementary event**: $1-(P(T=1)+P(T=2)) = 1-\frac{4}{36} = \frac{32}{36}$.

- **(ii) Probability a player scores 6, given they scored at least 3:** This is a **conditional probability** calculated as:

$$\begin{aligned}P(T=6 \mid T \geq 3) &= \frac{P(T=6 \cap T \geq 3)}{P(T \geq 3)}\\ &= \frac{P(T=6)}{P(T \geq 3)}\\ &= \frac{11/36}{32/36}\\ &= \frac{11}{32}\end{aligned}$$

3. Expected Score of a Game

The **expected value** $E(T)$ is calculated using the formula $E(T) = \sum t \cdot P(T=t)$:

$$\begin{aligned}E(T) &= 1\left(\frac{1}{36}\right)+2\left(\frac{3}{36}\right)+3\left(\frac{5}{36}\right)+4\left(\frac{7}{36}\right)+5\left(\frac{9}{36}\right)+6\left(\frac{11}{36}\right)\\ &= \frac{1+6+15+28+45+66}{36}\\ &= \frac{161}{36}\\ &\approx 4.4722\end{aligned}$$

Final Answers: (a) Table completed with values: $\frac{1}{36}, \frac{3}{36}, \frac{5}{36}, \frac{7}{36}, \frac{9}{36}, \frac{11}{36}$ (b) (i) $\frac{8}{9}$ (b) (ii) $\frac{11}{32}$ (c) $\frac{161}{36}$

MATHAI_11_HL_Summer_2021_Q9

Solution

1. Calculation of w for given values of z

The relationship between the complex numbers w and z is defined by the linear transformation $w = iz + 1$.

- **(i) For $z = 2i$:** Substitute $z = 2i$ into the equation:

$$\begin{aligned} w &= i(2i) + 1 \\ &= 2i^2 + 1 \\ &= 2(-1) + 1 \\ &= -2 + 1 \\ &= -1 \end{aligned}$$

  Thus, $\boxed{w = -1}$.

- **(ii) For $z = 1 + i$:** Substitute $z = 1 + i$ into the equation:

$$\begin{aligned} w &= i(1 + i) + 1 \\ &= i + i^2 + 1 \\ &= i - 1 + 1 \\ &= i \end{aligned}$$

  Thus, $\boxed{w = i}$.

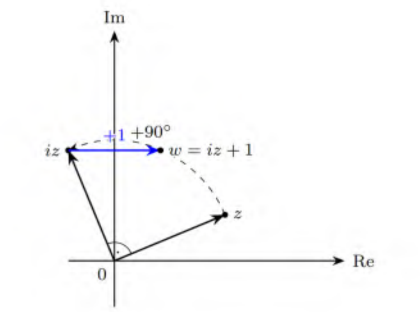
2. Geometric interpretation of the transformation

The transformation $w = iz + 1$ can be decomposed into two successive operations on the **Argand diagram**.

- **Step 1: Multiplication by i** Multiplying a complex number by i is equivalent to a **rotation** counter-clockwise by 90° (or $\pi/2$ radians) about the origin. This is because $i = e^{i\pi/2}$.
- **Step 2: Addition of 1** Adding 1 to the result is equivalent to a **translation** of 1 unit in the positive direction of the real axis (to the right).

Order of transformations:

1. A rotation of 90° counter-clockwise about the origin.
2. A translation by the vector $\binom{1}{0}$ (or 1 unit to the right).



3. Finding z when $w = 2 - i$

To find z, we rearrange the transformation equation $w = iz + 1$ to solve for z:

$$w - 1 = iz$$
$$z = \frac{w-1}{i}$$

Substitute $w = 2 - i$:

$$z = \frac{(2-i)-1}{i}$$
$$= \frac{1-i}{i}$$

To simplify, multiply the numerator and denominator by $-i$ (the conjugate of i):

$$z = \frac{(1-i)(-i)}{i(-i)}$$
$$= \frac{-i+i^2}{-i^2}$$
$$= \frac{-i-1}{-(-1)}$$
$$= \frac{-1-i}{1}$$
$$= -1 - i$$

$\boxed{z = -1 - i}$

MATHAI_11_HL_Summer_2021_Q10

Solution

To find a lower bound for the **Travelling Salesperson Problem** (TSP) using the **MST method**, we follow the steps outlined in the problem.

1. Part (a) (i): Prim's Algorithm on Subgraph (G - {A})

We delete vertex A and all its incident edges. The remaining vertices are {B, C, D, E, F}. We apply **Prim's algorithm** starting at vertex B.

- **Step 1:** Start at B. The available edges from B are BC (46), BD (58), BE (88), and BF (92). Select the shortest edge: **BC (46)**.
- **Step 2:** From {B, C}, the available edges to new vertices are BD (58), BE (88), BF (92) from B, and CD (87), CE (77), CF (66) from C. Select the shortest edge: **BD (58)**.
- **Step 3:** From {B, C, D}, the available edges to new vertices are BE (88), BF (92) from B; CE (77), CF (66) from C; and DE (23), DF (70) from D. Select the shortest edge: **DE (23)**.
- **Step 4:** From {B, C, D, E}, the available edges to the final vertex F are BF (92), CF (66), DF (70), and EF (47). Select the shortest edge: **EF (47)**.

The edges selected in order are:

$$BC, BD, DE, EF$$

The weight of this **Minimum Spanning Tree** (MST) is:

$$W_{\text{MST}} = 46 + 58 + 23 + 47 = 174 \,\text{min}$$

2. Part (a) (ii): Lower Bound for the Travelling Time

To find the lower bound for the total tour starting and finishing at A, we add the two shortest edges incident to the deleted vertex A to the weight of the MST of the remaining subgraph.

- The edges incident to A are: AB (55), AC (63), AD (79), AE (87), AF (93).
- The two shortest edges are **AB (55)** and **AC (63)**.

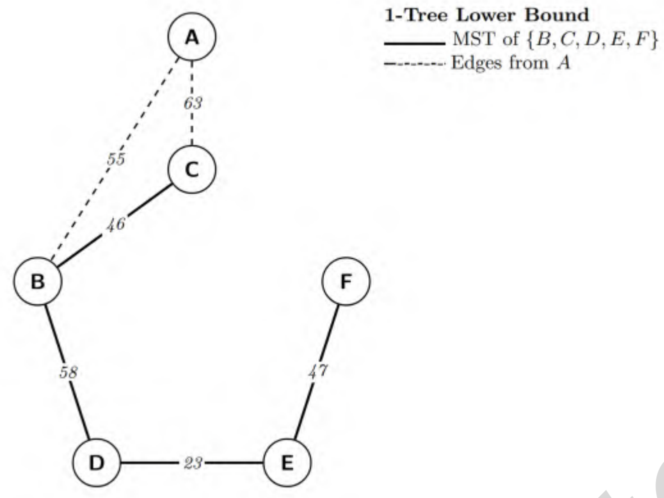
The lower bound is calculated as:

$$\begin{aligned}\text{Lower Bound} &= W_{\text{MST}} + \text{Shortest edge to A} + \text{Second shortest edge to A}\\ &= 174 + 55 + 63\\ &= 292 \,\text{min}\end{aligned}$$

$\boxed{292 \,\text{min}}$

3. Part (b): Improved Lower Bound

An improved lower bound can be found by repeating the process described in part (a) for each of the other vertices (B, C, D, E, and F) in turn. By deleting each vertex one by one, calculating the MST of the remaining subgraph, and adding the two shortest edges connecting the deleted vertex back to the subgraph, we obtain a set of potential lower bounds. The **best lower bound** is the maximum value among all these calculated bounds.


1-Tree Lower Bound
MST of $\{B, C, D, E, F\}$
Edges from A
A
B
C
D
E
F
63
55
46
58
23
47

MATHAI_11_HL_Summer_2021_Q11

Solution

1. Sampling Method

The restaurant manager checks all toys within a single delivered box. Since the sample is chosen based on what is easily available or already delivered, this is a form of **convenience sampling**. Alternatively, because the box represents a naturally occurring group within the production population, it can be classified as **cluster sampling**. In most introductory statistics contexts, checking a single delivered batch is identified as:

- **Convenience Sampling**

2. Null and Alternative Hypotheses

Let p represent the true proportion of faulty toys produced by the factory. The factory claims $p = 0.01$. The manager wants to test if the claim is reasonable given that 4 out of 200 toys were faulty (an observed proportion of 0.02). Since the observed value is higher than the claim, a **one-tailed test** is conducted to see if the proportion is actually greater than claimed.

- **Null Hypothesis** (H_0): $p = 0.01$
- **Alternative Hypothesis** (H_1): $p > 0.01$

3. Calculation of the p-value

Since the faults occur independently and there are two outcomes (faulty or not faulty), the number of faulty toys X follows a **Binomial distribution**:

$$X \sim B(n, p) = B(200, 0.01)$$

The **p-value** is the probability of observing a result as extreme as, or more extreme than, the observed data (4 faulty toys), assuming the null hypothesis is true:

$$P(X \geq 4) = 1 - P(X \leq 3)$$

Using the binomial cumulative distribution function:

$$\begin{aligned} P(X \leq 3) &= \sum_{k=0}^{3} \binom{200}{k} (0.01)^k (0.99)^{200-k} \\ &= \binom{200}{0}(0.99)^{200} + \binom{200}{1}(0.01)(0.99)^{199} + \binom{200}{2}(0.01)^2(0.99)^{198} + \binom{200}{3}(0.01)^3(0.99)^{197} \\ &\approx 0.1339796 + 0.2706659 + 0.2720334 + 0.1813556 \\ &\approx 0.8580345 \end{aligned}$$

Then, the p-value is:

$$\begin{aligned} p\text{ -value} &= 1 - 0.8580345 \\ &= 0.1419655 \end{aligned}$$

Rounding to three significant figures:

$$\boxed{0.142}$$

4. Conclusion of the Test

To state the conclusion, we compare the p-value to the **significance level** ($\alpha = 0.10$).

- **Comparison**: $0.142 > 0.10$
- **Reason**: Since the p-value is greater than the significance level, we fail to reject the null hypothesis.
- **Conclusion**: There is insufficient evidence at the 10% significance level to suggest that the proportion of faulty toys is greater than 1%. The factory's claim is considered reasonable.

MATHAI_11_HL_Summer_2021_Q12

Solution

1. Solving the Differential Equation

The rate of change of the volume V with respect to time t is given by the **separable differential equation**:

$$\frac{dV}{dt} = -k\sqrt{V}$$

- To solve for V, we separate the variables V and t:

$$\frac{1}{\sqrt{V}} dV = -k\,dt$$

- Integrating both sides:

$$\int V^{-1/2} dV = \int -k\,dt$$

$$2\sqrt{V} = -kt + C$$

- We use the initial condition provided: at $t = 0$, $V = 400\,\text{litres}$.

$$2\sqrt{400} = -k(0) + C$$

$$2(20) = C \implies C = 40$$

- Substituting C back into the general solution:

$$2\sqrt{V} = -kt + 40$$

2. Determining the Constant k

- We use the second condition: at $t = 10\,\text{minutes}$, $V = 324\,\text{litres}$.

$$2\sqrt{324} = -k(10) + 40$$

$$2(18) = -10k + 40$$

$$36 = -10k + 40$$

$$10k = 40 - 36$$

$$10k = 4 \implies k = 0.4$$

3. Deriving the Expression for V

- Substitute $k = 0.4$ into the equation $2\sqrt{V} = -kt + 40$:

$$2\sqrt{V} = -0.4t + 40$$

- Divide the entire equation by 2:

$$\sqrt{V} = 20 - 0.2t$$

- Since $0.2 = \frac{1}{5}$, we can write:

$$\sqrt{V} = 20 - \frac{t}{5}$$

- Squaring both sides yields the required expression:

$$V = \left(20 - \frac{t}{5}\right)^2$$

4. Time Taken for the Tank to Empty

- The tank is empty when the volume $V = 0$.

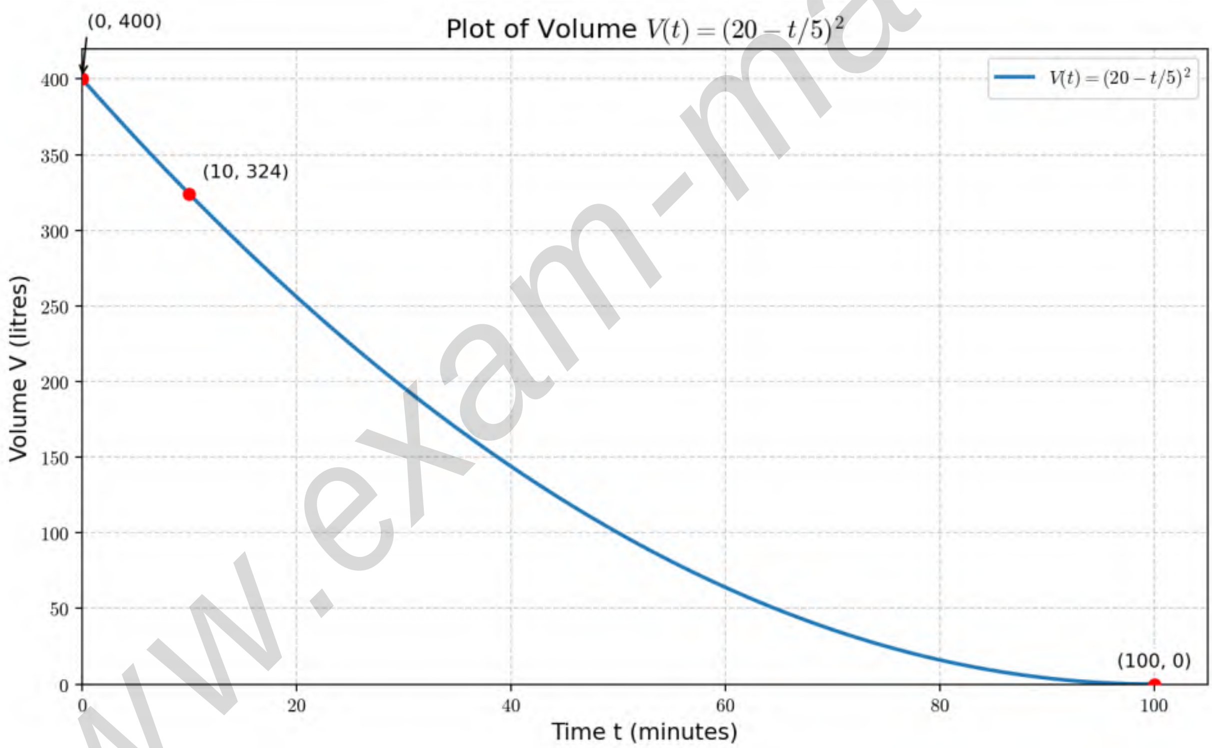
$$0 = \left(20 - \frac{t}{5}\right)^2$$

- Taking the square root of both sides:

$$0 = 20 - \frac{t}{5}$$

$$\frac{t}{5} = 20$$

$$t = 100\,\text{minutes}$$



(a) $V = \left(20 - \frac{t}{5}\right)^2$
(b) $\boxed{100\,\text{minutes}}$

MATHAI_11_HL_Summer_2021_Q13

Solution

1. Equation of the Submarine's Path

The submarine starts at the point $S(0.8, 1.3, -0.3)$ and travels in a straight line with a **direction vector** $\mathbf{d} = \begin{pmatrix} -2 \\ -3 \\ 1 \end{pmatrix}$. The **vector equation of a line** in 3D space is given by $\mathbf{r} = \mathbf{a} + \lambda\mathbf{d}$, where $\mathbf{a}$ is the position vector of a point on the line and λ is a scalar parameter.

Substituting the given values:

$$\mathbf{r} = \begin{pmatrix} 0.8 \\ 1.3 \\ -0.3 \end{pmatrix} + \lambda \begin{pmatrix} -2 \\ -3 \\ 1 \end{pmatrix}$$

2. Coordinates of Point P

The point P is where the submarine reaches the surface of the sea. In the given coordinate system, the z-direction is vertically upwards, and the ship is at the origin $O(0, 0, 0)$. Therefore, the surface of the sea corresponds to the xy-plane, where $z = 0$.

- **Step 1: Find the parameter λ at the surface.** From the z-component of the line equation:

  $$z = -0.3 + \lambda(1)$$

  Setting $z = 0$ to find the intersection with the surface:

  $$0 = -0.3 + \lambda \Longrightarrow \lambda = 0.3$$

- **Step 2: Calculate the x and y coordinates at $\lambda = 0.3$.**

  $$x = 0.8 + 0.3(-2) = 0.8 - 0.6 = 0.2$$
  $$y = 1.3 + 0.3(-3) = 1.3 - 0.9 = 0.4$$

  Thus, the coordinates of point P are $(0.2, 0.4, 0)$.

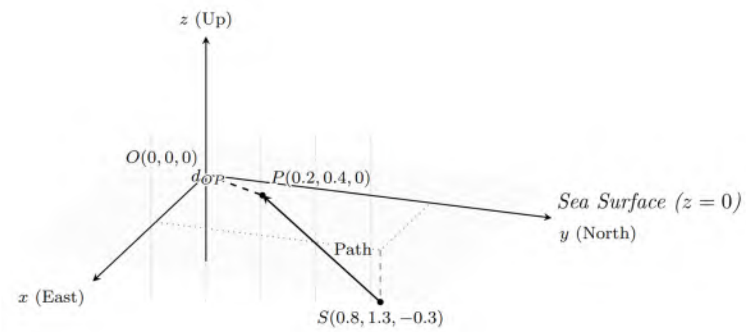
3. Distance OP

The distance OP is the magnitude of the position vector of point P relative to the origin $O(0, 0, 0)$. Using the **Euclidean distance** formula:

$$\begin{aligned} OP &= \sqrt{x^2 + y^2 + z^2} \\ &= \sqrt{(0.2)^2 + (0.4)^2 + (0)^2} \\ &= \sqrt{0.04 + 0.16} \\ &= \sqrt{0.2} \end{aligned}$$

Calculating the numerical value:

$$OP \approx 0.447213595499958\,\mathrm{km}$$



(a) $\mathbf{r} = \begin{pmatrix} 0.8 \\ 1.3 \\ -0.3 \end{pmatrix} + \lambda \begin{pmatrix} -2 \\ -3 \\ 1 \end{pmatrix}$

(b) (i) $P(0.2, 0.4, 0)$

(ii) $OP = \sqrt{0.2} \approx 0.447\,\text{km}$

MATHAI_11_HL_Summer_2021_Q14

Solution

Let X be the random variable representing the weight of a single apple. According to the problem, X follows a **normal distribution** with mean $\mu = 158\,\mathrm{g}$ and standard deviation $\sigma = 13\,\mathrm{g}$. A bag contains six apples. Let W be the random variable representing the total weight of a bag, where $W = X_1 + X_2 + X_3 + X_4 + X_5 + X_6$. We assume the weights of the apples are independent and identically distributed (i.i.d.).

1. Mean weight of a bag of apples The mean of the sum of independent random variables is the sum of their means. - $\mu_W = E[X_1 + X_2 + X_3 + X_4 + X_5 + X_6]$ - $\mu_W = 6 \cdot \mu$ - $\mu_W = 6 \cdot 158\,\mathrm{g} = 948\,\mathrm{g}$

$\boxed{948\,\mathrm{g}}$

2. Standard deviation of the weights of the bags For independent random variables, the **variance** of the sum is the sum of the variances. - $\sigma_W^2 = \mathrm{Var}\,(X_1 + X_2 + X_3 + X_4 + X_5 + X_6)$ - $\sigma_W^2 = 6 \cdot \sigma^2$ - $\sigma_W = \sqrt{6 \cdot \sigma^2} = \sigma\sqrt{6}$ - $\sigma_W = 13\sqrt{6}\,\mathrm{g}$ - $\sigma_W \approx 31.843366\,\mathrm{g}$

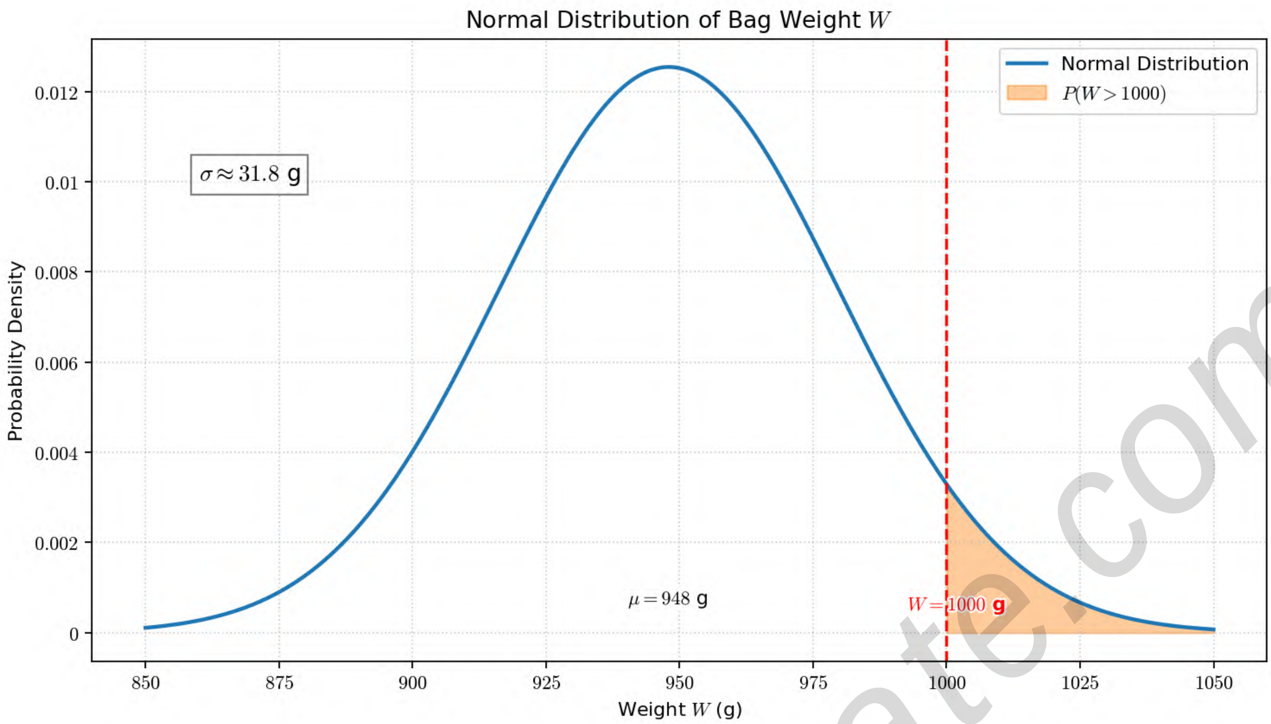
$\boxed{31.8\,\mathrm{g}}$ (to 3 significant figures)

3. Probability that a bag weighs more than $1\,\mathrm{kg}$ Since the weight of each apple is normally distributed, the total weight W also follows a normal distribution: $W \sim N(\mu_W, \sigma_W^2)$. We need to find $P(W > 1000\,\mathrm{g})$. - First, calculate the **z-score** for $w = 1000\,\mathrm{g}$:

$$\begin{aligned} z &= \frac{w - \mu_W}{\sigma_W} \\ &= \frac{1000 - 948}{13\sqrt{6}} \\ &= \frac{52}{13\sqrt{6}} \\ &= \frac{4}{\sqrt{6}} \approx 1.632993 \end{aligned}$$

- The probability is given by $P(Z > 1.632993)$, where $Z \sim N(0, 1)$. - Using the standard normal distribution table or a calculator:

$$\begin{aligned} P(W > 1000) &= 1 - \Phi(1.632993) \\ &\approx 1 - 0.948765 \\ &\approx 0.051235 \end{aligned}$$



$\boxed{0.0512}$ (to 3 significant figures)

MATHAI_11_HL_Summer_2021_Q15

Solution

The problem provides a **slope field** for the **differential equation**:

$$\frac{dy}{dx} = \sin(x+y)$$

defined over the domain $-4 \leq x \leq 5$ and $0 \leq y \leq 5$. We are asked to find the equations of lines L_1 and L_2 where the local minima and local maxima of the solution curves lie.

1. Condition for Local Extrema A local extremum (minimum or maximum) of a differentiable function $y(x)$ occurs where the first derivative is zero. From the given differential equation, we set:

$$\frac{dy}{dx} = 0 \Longrightarrow \sin(x+y) = 0$$

The sine function is zero when its argument is an integer multiple of π:

$$x + y = k\pi, \quad k \in \mathbb{Z}$$

Rearranging for y gives the general equation for the **isocline** of zero slope:

$$y = -x + k\pi$$

2. Identifying Local Minima (L_1) To distinguish between a local minimum and a local maximum, we examine the **second derivative** using the **chain rule**:

$$\frac{d^2y}{dx^2} = \frac{d}{dx}[\sin(x+y)]$$
$$= \cos(x+y) \cdot \left(1 + \frac{dy}{dx}\right)$$

At a point where $\frac{dy}{dx} = 0$, the second derivative simplifies to:

$$\frac{d^2y}{dx^2} = \cos(x+y)$$

For a local minimum, the **Second Derivative Test** requires $\frac{d^2y}{dx^2} > 0$.

- This occurs when $\cos(x+y) > 0$.
- Since we already established $x + y = k\pi$, we check the values of $\cos(k\pi) = (-1)^k$.
- For the result to be positive, k must be an odd integer (e.g., $k = -1, 1, 3, ...$).

Looking at the provided graph, the local minima for the curves passing through $(0, 1)$ and $(0, 3)$ occur in the region where x is negative (around $x \approx -1.5$ to -2).

- If $k = -1$, then $y = -x - \pi$. For $x \approx -2$, $y \approx 2 - 3.14 = -1.14$. This is outside the visible y-range ($0 \leq y \leq 5$).
- If $k = 1$, then $y = -x + \pi$. For $x \approx -2$, $y \approx 2 + 3.14 = 5.14$. This is at the top edge of the graph.

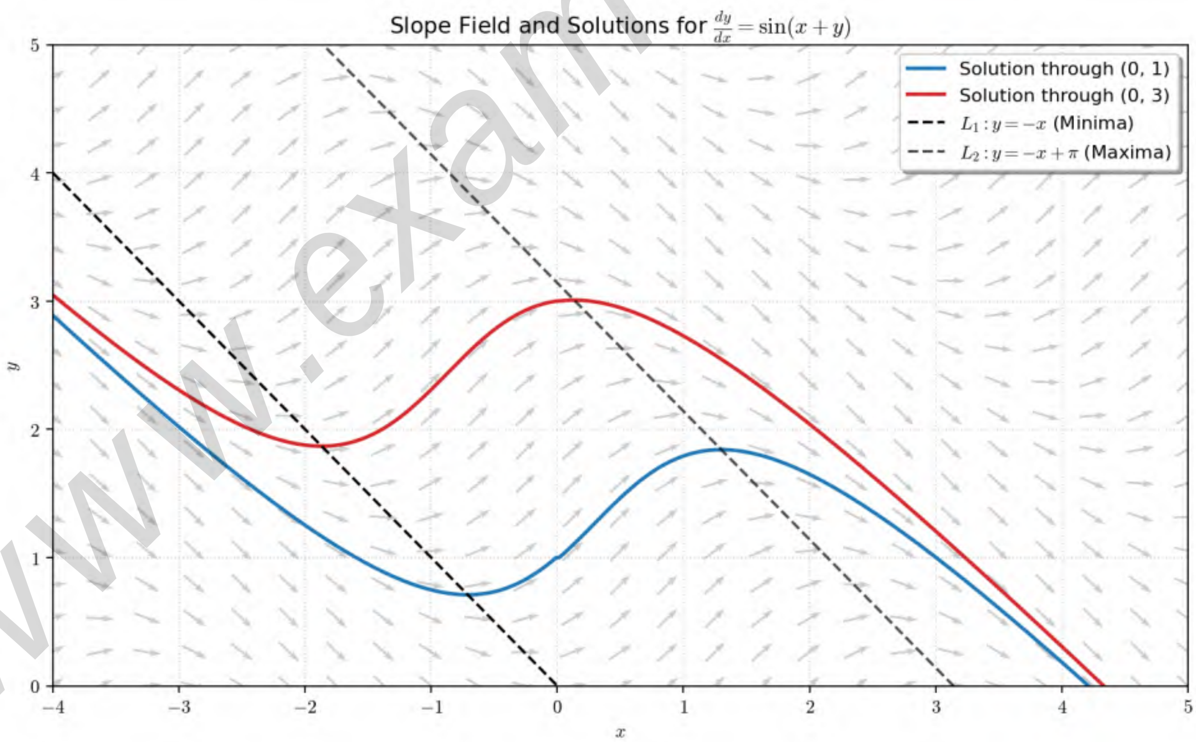
- However, observing the slope field segments, the horizontal segments (zero slope) forming the "troughs" of the curves align with the line where $x + y$ is a specific constant.
- Testing $k = -1$: $y = -x - \pi$.
- Testing $k = 1$: $y = -x + \pi$. The visual evidence shows the minima align on a line with a slope of -1. Given the positions of the troughs (e.g., near $(-2, 1)$ and $(-1, 2)$), the sum $x + y$ is approximately -1 or 2. Since the condition is $x + y = k\pi$, and $\cos(k\pi)$ must be positive for a minimum, k must be an even integer.
- Let $k = 0$: $y = -x$. Then $\cos(0) = 1 > 0$. This fits the visual data where $x + y \approx 0$ at the minima.

$$L_1 : y = -x$$

3. Identifying Local Maxima (L_2) For a local maximum, the Second Derivative Test requires $\frac{d^2y}{dx^2} < 0$.

- This occurs when $\cos(x + y) < 0$.
- For $x + y = k\pi$, this requires k to be an odd integer (e.g., $k = ..., -1, 1, 3, ...$).
- Looking at the graph, the local maxima occur in the region where x is positive (around $x \approx 1$ to 2).
- For the curve passing through $(0, 3)$, the maximum is at (x, y) such that $x + y \approx 3$.
- Setting $k = 1$: $y = -x + \pi$. Here $\cos(\pi) = -1 < 0$, which confirms a maximum.

$$L_2 : y = -x + \pi$$



Final Equations

(a) The line L_1 for local minima:

$$\boxed{y = -x}$$

(b) The line L_2 for local maxima:

$$\boxed{y = -x + \pi}$$

MATHAI_11_HL_Summer_2021_Q16

Solution

The problem involves a graph representation of a triangular prism. We are asked to determine the adjacency matrix of the graph and use it to find the number of paths of length 6 starting and ending at vertex A.

1. Construction of the Adjacency Matrix

The graph consists of six vertices: $\{A, B, C, D, E, F\}$. Based on the provided diagram, the connections (edges) are as follows:

- Vertex A is connected to B, C, D.
- Vertex B is connected to A, C, E.
- Vertex C is connected to A, B, F.
- Vertex D is connected to A, E, F.
- Vertex E is connected to B, D, F.
- Vertex F is connected to C, D, E.

The **adjacency matrix** M is a 6×6 matrix where the element $M_{ij} = 1$ if there is an edge between vertex i and vertex j, and 0 otherwise. Ordering the vertices as (A, B, C, D, E, F), we obtain:

$$M = (0\ 1\ 1\ 1\ 0\ 0; 1\ 0\ 1\ 0\ 1\ 0; 1\ 1\ 0\ 0\ 0\ 1; 1\ 0\ 0\ 0\ 1\ 1; 0\ 1\ 0\ 1\ 0\ 1; 0\ 0\ 1\ 1\ 1\ 0)$$

2. Calculating the Number of Walks of Length 6

A fundamental theorem in **graph theory** states that the number of walks of length k between vertex i and vertex j is given by the entry in the i-th row and j-th column of the matrix M^k. To find the number of ways to start at A and return to A in exactly 6 steps, we need to calculate the entry $(M^6)_{1,1}$.

- **Step 2.1: Calculate M^2**

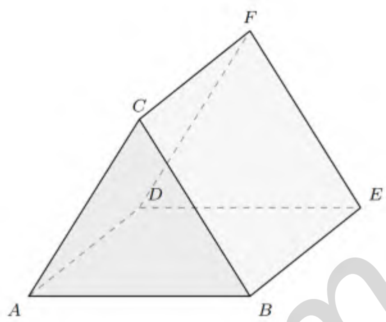
$$M^2 = (3\ 1\ 1\ 0\ 2\ 2; 1\ 3\ 1\ 2\ 0\ 2; 1\ 1\ 3\ 2\ 2\ 0; 0\ 2\ 2\ 3\ 1\ 1; 2\ 0\ 2\ 1\ 3\ 1; 2\ 2\ 0\ 1\ 1\ 3)$$

- **Step 2.2: Calculate $M^4 = M^2 \times M^2$** To find the first row of M^4, we multiply the first row of M^2 by each column of M^2:

$$(M^4)_{1,1} = 3(3) + 1(1) + 1(1) + 0(0) + 2(2) + 2(2) = 9 + 1 + 1 + 0 + 4 + 4 = 19$$
$$(M^4)_{1,2} = 3(1) + 1(3) + 1(1) + 0(2) + 2(0) + 2(2) = 3 + 3 + 1 + 0 + 0 + 4 = 11$$
$$(M^4)_{1,3} = 3(1) + 1(1) + 1(3) + 0(2) + 2(2) + 2(0) = 3 + 1 + 3 + 0 + 4 + 0 = 11$$
$$(M^4)_{1,4} = 3(0) + 1(2) + 1(2) + 0(3) + 2(1) + 2(1) = 0 + 2 + 2 + 0 + 2 + 2 = 8$$
$$(M^4)_{1,5} = 3(2) + 1(0) + 1(2) + 0(1) + 2(3) + 2(1) = 6 + 0 + 2 + 0 + 6 + 2 = 16$$
$$(M^4)_{1,6} = 3(2) + 1(2) + 1(0) + 0(1) + 2(1) + 2(3) = 6 + 2 + 0 + 0 + 2 + 6 = 16$$

- **Step 2.3: Calculate $(M^6)_{1,1}$** The number of walks of length 6 from A to A is the dot product of the first row of M^4 and the first column of M^2:

$$\begin{aligned}(M^6)_{1,1} &= (M^4)_{1,1}M^2_{1,1} + (M^4)_{1,2}M^2_{2,1} + (M^4)_{1,3}M^2_{3,1} + (M^4)_{1,4}M^2_{4,1} + (M^4)_{1,5}M^2_{5,1} + (M^4)_{1,6}M^2_{6,1} \\ &= 19(3) + 11(1) + 11(1) + 8(0) + 16(2) + 16(2) \\ &= 57 + 11 + 11 + 0 + 32 + 32 \\ &= 143\end{aligned}$$



(a) The adjacency matrix M is:

$$\mathbf{M} = (0\ 1\ 1\ 1\ 0\ 0; 1\ 0\ 1\ 0\ 1\ 0; 1\ 1\ 0\ 0\ 0\ 1; 1\ 0\ 0\ 0\ 1\ 1; 0\ 1\ 0\ 1\ 0\ 1; 0\ 0\ 1\ 1\ 1\ 0)$$

(b) The number of ways the ant can walk along exactly 6 edges to return to A is: $\boxed{143}$

MATHAI_11_HL_Summer_2021_Q17

Solution

1. Define the Translated Function

A **translation** of a function $f(x)$ by a vector $\binom{a}{b}$ results in a new function $g(x)$ defined by:

$$g(x) = f(x - a) + b$$

Given the parent function $f(x) = \ln x$, the translated function is:

$$g(x) = \ln(x - a) + b$$

The domain of this function is $x > a$.

2. Apply Boundary Conditions

The problem states that the graph of $g(x)$ passes through the points $(0, 1)$ and $(e^3, 1 + \ln 2)$. We substitute these coordinates into the equation for $g(x)$.

- For the point $(0, 1)$:

$$1 = \ln(0 - a) + b$$

$$1 = \ln(-a) + b \quad ...(1)$$

  Note that for $\ln(-a)$ to be defined, we must have $-a > 0$, which implies $a < 0$.

- For the point $(e^3, 1 + \ln 2)$:

$$1 + \ln 2 = \ln(e^3 - a) + b \quad ...(2)$$

3. Solve the System of Equations

Subtract equation (1) from equation (2) to eliminate the variable b:

$$(1 + \ln 2) - 1 = (\ln(e^3 - a) + b) - (\ln(-a) + b)$$
$$\ln 2 = \ln(e^3 - a) - \ln(-a)$$

Using the **logarithm quotient rule**, $\ln M - \ln N = \ln(M/N)$:

$$\ln 2 = \ln\left(\frac{e^3 - a}{-a}\right)$$

Since the natural logarithm is a one-to-one function, we equate the arguments:

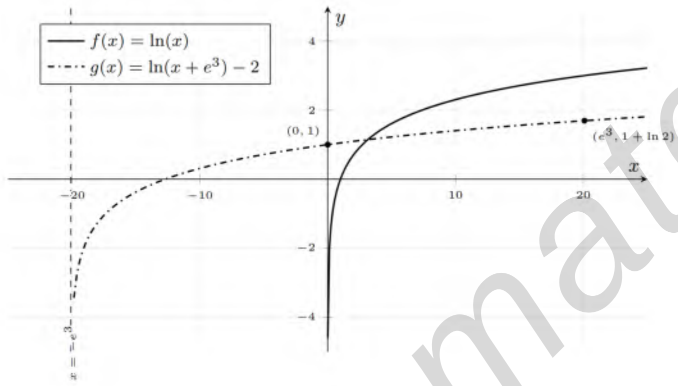
$$2 = \frac{e^3 - a}{-a}$$
$$-2a = e^3 - a$$
$$-a = e^3$$
$$a = -e^3$$

Now, substitute $a = -e^3$ back into equation (1) to find b:

$$1 = \ln(-(-e^3)) + b$$
$$1 = \ln(e^3) + b$$

Using the property $\ln(e^k) = k$:

$$1 = 3 + b$$
$$b = 1 - 3$$
$$b = -2$$



4. Final Values

The translation vector components are:

$$a = -e^3$$

$$b = -2$$

$\boxed{a = -e^3, b = -2}$